Answer keys for Sara Billey mid 1 Spring 2025

1(a).

$$4x_1 - 3x_2 + x_3 + 2x_4 = 6$$
$$-2x_3 + 4x_4 = 0$$
$$x_3 + x_5 = 1$$

1(b). 4 in row 1 column 1, -2 in row 2 column 3, 1 in row 3 column 3.
1(c). We need to apply Gaussian elimination to make it echelon:

$$\left[\begin{array}{ccccc|c} 4 & -3 & 1 & 2 & 0 & 6 \\ 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2/2\to R_3} \left[\begin{array}{ccccc|c} 4 & -3 & 1 & 2 & 0 & 6 \\ 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{array}\right]$$

Since all leading terms of the echelon form are not in the last column, the system has at least one solution, hence is consistent.

2. Here is the augmented matrix and Guass-Jordan elimination:

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 7 \\ 1 & 0 & 1 & 0 & 6 \\ 2 & 0 & 4 & 0 & 14 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 3 & 7 \\ 2 & 0 & 4 & 0 & 14 \end{array}\right]$$

$$\xrightarrow{R_3-2R_1\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow{R_3/2\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-R_3\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

The solution of the system is $x_4 = s$, $x_1 = 5$, $x_2 = 7 - 3s$, $x_3 = 1$:

$$(x_1, x_2, x_3, x_4) = (5, 7 - 3s, 1, s)$$

3. Such kind of questions is not involved in our class, but here is how you can solve it:
The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2s \\ 4t \\ 2t \\ 3r + t \end{bmatrix}$$

Since s, r can be any arbitrary numbers, x_1, x_4 will NOT appear in any equations. Then since $(x_2, x_3) = (4t, 2t)$, we only need an equation expressing such relation: $x_2 - 2x_3 = 0$.

In conclusion, any system equivalent (i.e., given by the non-zero multiples) to the equation $x_2 - 2x_3 = 0$ will have a solution set equal to the given span.

4(a).

$$\left[\begin{array}{cc|c} 2 & 1 & -4 \\ -1 & 3 & 9 \end{array}\right] \xrightarrow{R_2+R_1/2\to R_2} \left[\begin{array}{cc|c} 2 & 1 & -4 \\ 0 & 7/2 & 7 \end{array}\right] \xrightarrow{R_1/2\to R_1} \left[\begin{array}{cc|c} 1 & 1/2 & -2 \\ 0 & 7/2 & 7 \end{array}\right]$$

$$\xrightarrow{2R_2/7\to R_2} \left[\begin{array}{cc|c} 1 & 1/2 & -2 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1-R_2/2\to R_1} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right]$$

4(b). I'm not showing the pictures. The lines for the five matrices are

$$2x_1 + x_2 = -4\,,\ -x_1 + 3x_2 = 9$$
$$2x_1 + x_2 = -4\,, 7x_2/2 = 7$$
$$x_1 + x_2/2 = -2\,, 7x_2/2 = 7$$
$$x_1 + x_2/2 = -2\,, x_2 = 2$$
$$x_1 = -3\,, x_2 = 2$$

(The 2nd, 3rd, 4th graphs should look the same.) (You will end at a vertical line and a horizontal line.) (The intersection of the two lines will be the same for all the graphs.)

4(c). Input $x_1 = -3, x_2 = 2$ into the original equations:

$$2x_1 + x_2 = -4 \Rightarrow 2 \cdot (-3) + 2 = -4$$

$$-x_1 + 3x_2 = 9 \Rightarrow -(-3) + 3 \cdot 2 = 9$$

Both are satisfied.

5. To check if a vector (a, b, c, d) is a linear combination, we are solving the following equation system:

$$\begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -4 \\ 2 \\ 3 \\ 3 \end{bmatrix} x_2 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Its augmented matrix and echelon form are

$$\left[\begin{array}{cc|c} 3 & -4 & a \\ 1 & 2 & b \\ -2 & 3 & c \\ 1 & 3 & d \end{array}\right] \xrightarrow{R_2 - R_1/3 \to R_2} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 10/3 & b - a/3 \\ -2 & 3 & c \\ 1 & 3 & d \end{array}\right] \xrightarrow{R_3 + 2R_1/3 \to R_3} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 10/3 & b - a/3 \\ 0 & 1/3 & c + 2a/3 \\ 1 & 3 & d \end{array}\right]$$

$$\xrightarrow{R_4 - R_1/3 \to R_4} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 10/3 & b - a/3 \\ 0 & 1/3 & c + 2a/3 \\ 0 & 13/3 & d - a/3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 1/3 & c + 2a/3 \\ 0 & 10/3 & b - a/3 \\ 0 & 13/3 & d - a/3 \end{array}\right] \xrightarrow{R_3 - 10R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 1/3 & c + 2a/3 \\ 0 & 0 & b - 10c - 7a \\ 0 & 13/3 & d - a/3 \end{array}\right]$$

$$\xrightarrow{R_4 - 13R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 3 & -4 & a \\ 0 & 1/3 & c + 2a/3 \\ 0 & 0 & b - 10c - 7a \\ 0 & 0 & d - 13c - 9a \end{array}\right]$$

If we want (a, b, c, d) NOT to be a linear combination, we only need to make at least one of $b - 10c - 7a$ and $d - 13c - 9a$ nonzero. For example, we can choose $(a, b, c, d) = (0, 0, 0, 1)$.

6(a).

$$p + n + d = 52$$

$$p + 5n + 10d = 242$$

$$2.5p + 5n + 2.25d = 143$$

6(b). The first condition gives $d = u + 35$, which is $u - d = -35$. The second condition gives $u = 40\%(u + d)$, which is $0.6u - 0.4d = 0$.

The equation system is

$$u - d = -35$$

$$0.6u - 0.4d = 0$$